TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 5, Pages 2073–2090
S 0002-9947(00)02757-4
Article electronically published on December 29, 2000

A PRODUCT FORMULA FOR SPHERICAL REPRESENTATIONS OF A GROUP OF AUTOMORPHISMS OF A HOMOGENEOUS TREE, II

DONALD I. CARTWRIGHT AND GABRIELLA KUHN

ABSTRACT. Let $G = \mathrm{Aut}(T)$ be the group of automorphisms of a homogeneous tree T and let π be the tensor product of two spherical irreducible unitary representations of G. We complete the explicit decomposition of π commenced in part I of this paper, by describing the discrete series representations of G which appear as subrepresentations of π.

1. INTRODUCTION AND NOTATION

Let G be the group of automorphisms of a homogeneous tree T. We fix a vertex o of T, and let $K = \{g \in G : go = o\}$. We consider the tensor product π of two spherical irreducible unitary representations of $G = \mathrm{Aut}(T)$. As with any continuous unitary representation of a type I group, π can be written in an essentially unique way as a direct integral $\int_{\hat{G}} \sigma \, dm(\sigma)$. The representation space H_π of π can be decomposed as an orthogonal direct sum

$$H_\pi = H_1 + H_2, \tag{1.1}$$

of π-invariant subspaces H_1 and H_2, where H_1 is the closed linear span in H_π of the set of vectors $\pi(g)\xi$, where $g \in G$ and where ξ is K-invariant. The H_1 component π_1 of π was completely described in [1]. The H_2 component π_2 of π must be a direct sum

$$\pi_2 = \sum_k m_k \sigma_k, \tag{1.2}$$

over the distinct discrete series representations σ_k of G, where $m_k \in \{0, 1, \dots, \infty\}$ for each k. In the present paper we describe π_2 explicitly.

The discrete series representations of G were first described by Ol'shanskii [4]. The book [2] provides a clear exposition and re-working of [4]. Let us give a quick summary here.

A finite subtree $\mathfrak{x}$ of T is called *complete* if, for each vertex x of $\mathfrak{x}$, either all $q+1$ neighbors of x in T lie in $\mathfrak{x}$ or exactly one neighbor of x in T lies in $\mathfrak{x}$. According to these two possibilities, we say that x is an *interior point* or *boundary point* of $\mathfrak{x}$, and write $x \in \mathrm{Int}(\mathfrak{x})$ and $x \in \partial\mathfrak{x}$, respectively. Given such a subtree $\mathfrak{x}$, let

Received by the editors February 8, 2000.
2000 *Mathematics Subject Classification.* Primary 20E08, 20C15; Secondary 20C30.
Key words and phrases. Homogeneous tree, spherical representation.

$K(\mathfrak{x}) = \{g \in G : gx = x \text{ for all } x \in \mathfrak{x}\}$ and $\tilde{K}(\mathfrak{x}) = \{g \in G : g\mathfrak{x} = \mathfrak{x}\}$. If π is an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$, let

$$\mathcal{H}_{\mathfrak{x}} = \{\xi \in \mathcal{H} : \pi(g)\xi = \xi \text{ for all } g \in K(\mathfrak{x})\},$$

and let

$$P_{\mathfrak{x}} = \frac{1}{m(K(\mathfrak{x}))} \int_{K(\mathfrak{x})} \pi(g)\, dg,$$

the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_{\mathfrak{x}}$. Here m and dg refer to the left (and right) Haar measure on G, normalized by requiring that $m(K) = 1$. It turns out that $P_{\mathfrak{x}} \neq 0$ for some $\mathfrak{x}$, and if we choose $\mathfrak{x}$ so that the number $|\mathfrak{x}|$ of vertices of $\mathfrak{x}$ is minimal, then $\mathfrak{x}$ is called a *minimal subtree* for π. Note that if $g \in G$, then $K(g\mathfrak{x}) = gK(\mathfrak{x})g^{-1}$ and $P_{g\mathfrak{x}} = \pi(g)P_{\mathfrak{x}}\pi(g^{-1})$. If $\mathfrak{x}$ is a minimal subtree for π, then the other minimal subtrees $\mathfrak{x}'$ for π are precisely the subtrees $g\mathfrak{x}$, i.e., the subtrees in the G-orbit $[\mathfrak{x}] = \{g\mathfrak{x} : g \in G\}$ of $\mathfrak{x}$ [2, Cor. III.3.4].

When π has a minimal subtree consisting of only one vertex, it is spherical. In the next smallest case, $\mathfrak{x}$ consists of two neighboring vertices (and the edge joining them). Up to equivalence, there are precisely two irreducible unitary representations of G with such a minimal subtree; they are called the *special* representations of G (see [2, §III.2]). All irreducible unitary representations of G with minimal subtree $\mathfrak{x}$ satisfying $|\mathfrak{x}| > 2$ are equivalent to certain induced representations; one starts with an irreducible representation σ_0 of the finite group $\mathrm{Aut}(\mathfrak{x})$ on a (finite dimensional) Hilbert space $\mathcal{H}_{\sigma_0}$ such that, for each complete subtree $\mathfrak{y} \subsetneqq \mathfrak{x}$, $\sigma_0(g)\xi = \xi$ for all $g \in K_{\mathfrak{x}}(\mathfrak{y}) = \{g \in \mathrm{Aut}(\mathfrak{x}) : gy = y \text{ for all } y \in \mathfrak{y}\}$ holds for no nonzero $\xi \in \mathcal{H}_{\sigma_0}$. Let σ be the representation of $\tilde{K}(\mathfrak{x})$ obtained as the composition of the natural surjection $\tilde{K}(\mathfrak{x}) \to \mathrm{Aut}(\mathfrak{x})$ (the kernel of which is $K(\mathfrak{x})$) and σ_0. The set of all equivalence classes of representations σ of $\tilde{K}(\mathfrak{x})$ obtained in this way is denoted by $(\tilde{K}(\mathfrak{x}))^\wedge_0$. One then forms the induced representation $\mathrm{Ind}(\sigma)$ of G. It is shown in [2, Theorem III.3.14] that $\mathrm{Ind}(\sigma)$ is irreducible, and that $\sigma \mapsto \mathrm{Ind}(\sigma)$ induces a bijection from $(\tilde{K}(\mathfrak{x}))^\wedge_0$ to the set of equivalence classes of irreducible unitary representations of G with minimal subtree $\mathfrak{x}$.

The principal result of this paper is the following:

Theorem. *Let π be the tensor product of two spherical irreducible unitary representations of G. Then the irreducible unitary representations of G appearing in the direct sum* (1.2) *occur with multiplicity 1, and have minimal trees of one of the types in Figure 1. Moreover,*

(a) *both special representations occur;*
(b) *exactly two representations occur with the minimal subtree of Figure 1(b), unless $q = 2$, when there is only one;*
(c) *for each $r \geq 1$, exactly two representations occur with the minimal subtree of Figure 1(c_r).*

In (b) and (c), the specific representations occurring are described in Sections 3 and 4 below.

Notice that the decomposition (1.2) does not depend on the particular parameters s_1, s_2, which may correspond to either the principal or complementary series.

Our theorem should be compared with the the corresponding result for $SL(2, \mathbb{R})$. While here a rather thin part of the discrete series occurs in (1.2), in the $SL(2, \mathbb{R})$ case half of the discrete series occurs in the corresponding decomposition (see [5]).

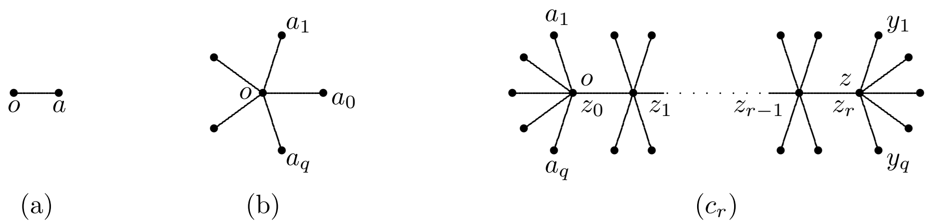


FIGURE 1.

In order to cover both the principal and complementary series simultaneously, let us introduce some more notation. Recall that (assuming $s^2 \neq q, 1/q$), $J_s : \mathcal{K}(\Omega) \to \mathcal{K}(\Omega)$ is a linear bijection and that $J_s \circ \pi^s(g) = \pi^{s^{-1}}(g) \circ J_s$ for all $g \in G$. For $x \in T$, $\xi_x \in \mathcal{K}(\Omega)$ was defined before Lemma 5.3 in [1]. We shall again here need the fact that

$$J_s \xi_x = j_n(s) \xi_x \quad \text{if } n = d(o, x), \tag{1.3}$$

where $j_0(s) = 1$ and $j_n(s) = s^{2(n-1)}(qs^2 - 1)/(q - s^2)$ for $s \geq 1$. If $s^2 \in (1/q, q)$, the inner product $\langle \cdot, \cdot \rangle_s$ on $\mathcal{K}(\Omega)$ is defined by $\langle f_1, f_2 \rangle_s = \langle f_1, J_s(f_2) \rangle$, where $\langle f_1, f_2 \rangle = \int_\Omega f_1 \bar{f}_2 \, d\nu_o$. Let I denote the identity operator on $\mathcal{K}(\Omega)$, and set

$$\tilde{J}_s = \begin{cases} J_s & \text{if } s^2 \in (1/q, q), \\ I & \text{if } |s| = 1. \end{cases}$$

Let $\langle \cdot, \cdot \rangle_s = \langle \cdot, \cdot \rangle$ if $|s| = 1$. Then $\langle f_1, f_2 \rangle_s = \langle f_1, \tilde{J}_s f_2 \rangle$ for all $f_1, f_2 \in \mathcal{K}(\Omega)$ if s is a parameter corresponding to either the principal or complementary series. Moreover, $\tilde{J}_s \circ \pi^s(g) = \pi^{\tilde{s}}(g) \circ \tilde{J}_s$ for all $g \in G$, where

$$\tilde{s} = \begin{cases} s^{-1} & \text{if } s^2 \in (1/q, q), \\ s & \text{if } |s| = 1. \end{cases}$$

If s_1, s_2 are parameters corresponding to either the principal or complementary series, let

$$\tilde{J}_{s_1, s_2} = \tilde{J}_{s_1} \otimes \tilde{J}_{s_2} : \mathcal{K}(\Omega \times \Omega) \to \mathcal{K}(\Omega \times \Omega).$$

Then

$$\tilde{J}_{s_1, s_2} \circ \pi^{s_1, s_2}(g) = \pi^{\tilde{s}_1, \tilde{s}_2}(g) \circ \tilde{J}_{s_1, s_2},$$

where $\pi^{s_1, s_2}(g) = \pi^{s_1}(g) \otimes \pi^{s_2}(g)$. Also, $\pi^{s_1, s_2}(g)$ preserves the inner product

$$\langle F_1, F_2 \rangle_{s_1, s_2} = \langle F_1, \tilde{J}_{s_1, s_2}(F_2) \rangle',$$

where $\langle F_1, F_2 \rangle' = \int_{\Omega \times \Omega} F_1 \overline{F_2} \, d(\nu_o \times \nu_o)$. We shall also simply write $\pi(g)$ in place of $\pi^{s_1, s_2}(g)$, when s_1, s_2 are understood. The representation space $H_\pi = \mathcal{H}_{s_1, s_2}$ of π is the completion of $\mathcal{K}(\Omega \times \Omega)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{s_1, s_2}$.

The first author would like to thank Arun Ram for useful conversations concerning representations of the symmetric group.

2. The possible minimal subtrees

We start by describing, for each finite complete subtree $\mathfrak{x}$ of T,

$$\mathcal{H}'_{\mathfrak{x}} = \{\xi \in \mathcal{H}_{s_1,s_2} : \ (1)\ P_{\mathfrak{x}}\xi = \xi, \ \text{and}\ (2)\ P_{\mathfrak{y}}\xi = 0 \text{ for all } \mathfrak{y} \subsetneqq \mathfrak{x}\},$$

where the $\mathfrak{y}$'s here are complete subtrees of T. Note that if $g \in G$, then $\mathcal{H}'_{g(\mathfrak{x})} = \pi(g)\big(\mathcal{H}'_{\mathfrak{x}}\big)$, and so we may assume that $o \in \mathfrak{x}$. If $|\mathfrak{x}| > 2$, then $\mathfrak{x}$ has interior points, and in this case we may suppose that o is one of them.

Proposition 2.1. *Suppose that $|\mathfrak{x}| > 2$ and that $o \in Int(\mathfrak{x})$. Then $\mathcal{H}'_{\mathfrak{x}}$ is a finite dimensional subspace of $\mathcal{K}(\Omega \times \Omega)$. Assuming that $\mathcal{H}'_{\mathfrak{x}} \neq \{0\}$, then either*

(i) *$\mathfrak{x}$ is as in Figure 1(b); in this case $\mathcal{H}'_{\mathfrak{x}}$ consists of the functions*

$$\sum_{\substack{i,j=0\\ i\neq j}}^{q} c_{i,j}\xi_{a_i} \otimes \xi_{a_j}, \tag{2.1}$$

where the coefficients $c_{i,j}$ satisfy

$$\sum_{\substack{j=0\\ j\neq i}}^{q} c_{i,j} = 0 \quad \text{for each } i, \text{ and } \quad \sum_{\substack{i=0\\ i\neq j}}^{q} c_{i,j} = 0 \quad \text{for each } j, \tag{2.2}$$

or

(ii) *replacing $\mathfrak{x}$ by $g\mathfrak{x}$ for some $g \in G$, if necessary, $\mathfrak{x}$ is as in Figure 1(c_r); in this case $\mathcal{H}'_{\mathfrak{x}}$ consists of the functions*

$$\sum_{i,j=1}^{q} c_{i,j}\xi_{a_i} \otimes \xi_{y_j} + \sum_{i,j=1}^{q} d_{i,j}\xi_{y_j} \otimes \xi_{a_i}, \tag{2.3}$$

where the coefficients $c_{i,j}$ and $d_{i,j}$ satisfy

$$\sum_{j=1}^{q} c_{i,j} = 0 = \sum_{j=1}^{q} d_{i,j} \quad \text{for each } i, \text{ and } \quad \sum_{i=1}^{q} c_{i,j} = 0 = \sum_{i=1}^{q} d_{i,j} \quad \text{for each } j. \tag{2.4}$$

Proof. For each $v \in \partial\mathfrak{x}$ and $n \geq |v|$ (where $|v| = d(o,v)$), let

$$D_{v,n} = \{(\omega_1,\omega_2) \in \Omega_o(v) \times \Omega_o(v) : \omega_1 \neq \omega_2 \text{ and } d(o,\omega_1 \wedge \omega_2) = n\}; \tag{2.5}$$

we also let

$$\Delta_v = \{(\omega_1,\omega_2) \in \Omega_o(v) \times \Omega_o(v) : \omega_1 = \omega_2\}$$

for each $v \in \partial\mathfrak{x}$. Then

$$\Omega \times \Omega = \bigcup_{v \in \partial\mathfrak{x}} \Delta_v \cup \bigcup_{\substack{v\in\partial\mathfrak{x}\\ n\geq|v|}} D_{v,n} \cup \bigcup_{\substack{v,w\in\partial\mathfrak{x}\\ v\neq w}} \Omega_o(v) \times \Omega_o(w),$$

a disjoint union. Let $F \in \mathcal{K}(\Omega \times \Omega)$ satisfy $P_{\mathfrak{x}}F = F$. As $o \in \mathfrak{x}$, this simply means that $F(g^{-1}\omega_1, g^{-1}\omega_2) = F(\omega_1,\omega_2)$ for all $g \in K(\mathfrak{x})$ and all $\omega_1,\omega_2 \in \Omega$. It follows that F is constant on each of the above sets Δ_v, $D_{v,n}$ and $\Omega_o(v) \times \Omega_o(w)$. Hence we can write

$$F = \sum_{v\in\partial\mathfrak{x}} a_v \mathbf{1}_{\Delta_v} + \sum_{\substack{v\in\partial\mathfrak{x}\\ n\geq|v|}} b_{v,n}\mathbf{1}_{D_{v,n}} + \sum_{\substack{v,w\in\partial\mathfrak{x}\\ v\neq w}} c_{v,w}\mathbf{1}_{\Omega_o(v)\times\Omega_o(w)}. \tag{2.6}$$

Moreover, the fact that $F \in \mathcal{K}(\Omega \times \Omega)$ implies that for each $v \in \partial\mathfrak{x}$ there is an n_v such that $b_{v,n} = a_v$ for all $n \geq n_v$. Conversely, any function F of the form (2.6) which satisfies this last condition is in $\mathcal{K}(\Omega \times \Omega)$ and satisfies $P_\mathfrak{x} F = F$. Thus each $F \in \mathcal{K}(\Omega \times \Omega)$ satisfying $P_\mathfrak{x} F = F$ can be written in the form

$$F = \sum_{v \in \partial\mathfrak{x}} \Big\{ \sum_{n=|v|}^{M} b_{v,n} \mathbf{1}_{D_{v,n}} \ + \ a_v \mathbf{1}_{D'_{v,M}} \Big\} + \sum_{\substack{v,w \in \partial\mathfrak{x} \\ v \neq w}} c_{v,w} \mathbf{1}_{\Omega_o(v) \times \Omega_o(w)} \tag{2.7}$$

for some integer $M \geq \max\{|v| : v \in \partial\mathfrak{x}\}$. Here

(2.8)

$$D'_{v,M} = \{(\omega_1, \omega_2) \in \Omega_o(v) \times \Omega_o(v) : \omega_1 = \omega_2, \text{ or } \omega_1 \neq \omega_2 \text{ and } d(o, \omega_1 \wedge \omega_2) > M\}.$$

Suppose $\xi \in \mathcal{H}_{s_1,s_2}$ and $P_\mathfrak{x}\xi = \xi$. Let us write write down a sequence (F_N) in $\mathcal{K}(\Omega \times \Omega)$ such that $P_\mathfrak{x} F_N = F_N$ for each N and such that, for each $f \in \mathcal{K}(\Omega \times \Omega)$, $\langle F_N, f\rangle_{s_1,s_2} \to \langle \xi, f\rangle_{s_1,s_2}$ as $N \to \infty$. If $\xi \in \mathcal{H}_{s_1,s_2}$, and if $F \in \mathcal{K}(\Omega \times \Omega)$, it is convenient to use the notation

$$\langle \xi, F\rangle' = \langle \xi, \tilde{J}^{-1}_{s_1,s_2} F\rangle_{s_1,s_2}. \tag{2.9}$$

For $v, w \in \partial\mathfrak{x}$, $n \geq |v|$ and $N \geq \max\{|v| : v \in \partial\mathfrak{x}\}$, let

$$\begin{aligned} b_{v,n} &= \langle \xi, \mathbf{1}_{D_{v,n}}\rangle' / (\nu_o \times \nu_o)(D_{v,n}), \\ c_{v,w} &= \langle \xi, \mathbf{1}_{\Omega_o(v) \times \Omega_o(w)}\rangle' / (\nu_o \times \nu_o)(\Omega_o(v) \times \Omega_o(w)), \text{ and} \\ \delta_{v,N} &= \langle \xi, \mathbf{1}_{D'_{v,N}}\rangle' / (\nu_o \times \nu_o)(D'_{v,N}). \end{aligned}$$

Form

$$F_N = \sum_{v \in \partial\mathfrak{x}} \Big\{ \sum_{n=|v|}^{N} b_{v,n} \mathbf{1}_{D_{v,n}} \ + \ \delta_{v,N} \mathbf{1}_{D'_{v,N}} \Big\} + \sum_{\substack{v,w \in \partial\mathfrak{x} \\ v \neq w}} c_{v,w} \mathbf{1}_{\Omega_o(v) \times \Omega_o(w)}. \tag{2.10}$$

Then it is routine to check that if F is as in (2.7), then

$$\langle F_N, F\rangle' = \langle \xi, F\rangle' \quad \text{once } N \geq M.$$

It follows that $\langle F_N, f\rangle_{s_1,s_2} \to \langle \xi, f\rangle_{s_1,s_2}$ as $N \to \infty$ if $f \in \mathcal{K}(\Omega \times \Omega)$ satisfies $P_\mathfrak{x} f = f$. To see that this holds for arbitrary $f \in \mathcal{K}(\Omega \times \Omega)$, first note that, because $o \in \mathfrak{x}$,

$$\big(\pi^{s_1,s_2}(g) f\big)(\omega_1, \omega_2) = f(g^{-1}\omega_1, g^{-1}\omega_2) = \big(\pi^{\tilde{s}_1,\tilde{s}_2}(g) f\big)(\omega_1, \omega_2)$$

for all $g \in K(\mathfrak{x})$ and $\omega_1, \omega_2 \in \Omega$, and hence

$$\pi^{s_1,s_2}(g)\big(\tilde{J}^{-1}_{s_1,s_2}(f)\big) = \tilde{J}^{-1}_{s_1,s_2}\big(\pi^{\tilde{s}_1,\tilde{s}_2}(g)(f)\big) = \tilde{J}^{-1}_{s_1,s_2}\big(\pi^{s_1,s_2}(g)(f)\big). \tag{2.11}$$

Thus $P_\mathfrak{x} \tilde{J}^{-1}_{s_1,s_2} f = \tilde{J}^{-1}_{s_1,s_2}\big(P_\mathfrak{x} f\big)$. Hence for any $f \in \mathcal{K}(\Omega \times \Omega)$,

$$\langle \xi, f\rangle' = \langle P_\mathfrak{x}\xi, f\rangle' = \langle \xi, P_\mathfrak{x} f\rangle',$$

because $P_\mathfrak{x}$ is Hermitian with respect to $\langle \cdot, \cdot\rangle_{s_1,s_2}$. Note also that the projection $P_\mathfrak{x}$ leaves $\mathcal{K}(\Omega \times \Omega)$ invariant. For the same reasons, $\langle F_N, f\rangle' = \langle P_\mathfrak{x} F_N, f\rangle' = \langle F_N, P_\mathfrak{x} f\rangle'$. It follows that $\langle F_N, f\rangle' \to \langle \xi, f\rangle'$ as $N \to \infty$ for any $f \in \mathcal{K}(\Omega \times \Omega)$.

We next show that if $\xi \in \mathcal{H}'_\mathfrak{x}$, then all the coefficients $b_{v,n}$ and $\delta_{v,N}$ in (2.10) are zero. If $v \in \partial\mathfrak{x}$, there is clearly a complete subtree $\mathfrak{y} \subsetneq \mathfrak{x}$ containing o and v (because $|\mathfrak{x}| > 2$). It follows from (2.11) that, if $n \geq |v|$,

$$\pi^{s_1,s_2}(g)\big(\tilde{J}^{-1}_{s_1,s_2}(\mathbf{1}_{D_{v,n}})\big) = \tilde{J}^{-1}_{s_1,s_2}\big(\pi^{\tilde{s}_1,\tilde{s}_2}(g)(\mathbf{1}_{D_{v,n}})\big) = \tilde{J}^{-1}_{s_1,s_2}(\mathbf{1}_{D_{v,n}})$$

for all $g \in K(\mathfrak{y})$, and so $P_{\mathfrak{y}}\big(\tilde{J}^{-1}_{s_1,s_2}(\mathbf{1}_{D_{v,n}})\big) = \tilde{J}^{-1}_{s_1,s_2}(\mathbf{1}_{D_{v,n}})$. Thus

$$\begin{aligned}0 &= \langle P_{\mathfrak{y}}\xi, \mathbf{1}_{D_{v,n}}\rangle' \\ &= \langle \xi, P_{\mathfrak{y}}\big(\tilde{J}^{-1}_{s_1,s_2}\mathbf{1}_{D_{v,n}}\big)\rangle_{s_1,s_2} \\ &= \langle \xi, \tilde{J}^{-1}_{s_1,s_2}\mathbf{1}_{D_{v,n}}\rangle_{s_1,s_2} \\ &= (\nu_o \times \nu_o)(D_{v,n})b_{v,n},\end{aligned}$$

so that $b_{v,n} = 0$. Similarly $P_{\mathfrak{y}}\big(\tilde{J}^{-1}_{s_1,s_2}(\mathbf{1}_{D'_{v,N}})\big) = \tilde{J}^{-1}_{s_1,s_2}(\mathbf{1}_{D'_{v,N}})$, and so $\delta_{v,N} = 0$. Hence F_N, defined in (2.10), equals

$$F = \sum_{\substack{v,w\in\partial\mathfrak{x} \\ v\neq w}} c_{v,w}\mathbf{1}_{\Omega_o(v)\times\Omega_o(w)} \in \mathcal{K}(\Omega\times\Omega). \tag{2.12}$$

But as $\langle F_N, f\rangle' \to \langle \xi, f\rangle'$, we see that $\langle F, f\rangle' = \langle \xi, f\rangle'$ for each $f \in \mathcal{K}(\Omega\times\Omega)$, and so $\xi = F \in \mathcal{K}(\Omega\times\Omega)$.

If $X \subset T$ is a finite set of vertices, let $\mathfrak{y}(X)$ denote the smallest complete subtree of T containing X. As we have already noted, each projection $P_{\mathfrak{y}}$ leaves $\mathcal{K}(\Omega\times\Omega)$ invariant, and when $o \in \mathfrak{y}$ and $f \in \mathcal{K}(\Omega\times\Omega)$ we have

$$(P_{\mathfrak{y}}f)(\omega_1,\omega_2) = \frac{1}{m(K(\mathfrak{y}))}\int_{K(\mathfrak{y})} f(g^{-1}\omega_1, g^{-1}\omega_2)\,dg.$$

We apply this formula to $f = F$, where $F \in \mathcal{H}'_{\mathfrak{x}}$ is given by (2.12). Observe that if $v, w \in \partial\mathfrak{x}$ are distinct, and if $\mathfrak{y} = \mathfrak{y}(\{o,v,w\})$ is a proper subtree of $\mathfrak{x}$, then $c_{v,w} = 0$. For we can choose $(\omega_1,\omega_2) \in \Omega_o(v)\times\Omega_o(w)$, and then $(g^{-1}\omega_1, g^{-1}\omega_2) \in \Omega_o(v)\times\Omega_o(w)$ for all $g \in K(\mathfrak{y})$. Hence $F(g^{-1}\omega_1, g^{-1}\omega_2) = c_{v,w}$ for all $g \in K(\mathfrak{y})$, so $0 = (P_{\mathfrak{y}}F)(\omega_1,\omega_2) = c_{v,w}$.

Assuming $\mathcal{H}'_{\mathfrak{x}} \neq \{0\}$, choose a nonzero $F \in \mathcal{H}'_{\mathfrak{x}}$. We know that F has the form (2.12), and as $F \neq 0$, we must have distinct $v, w \in \partial\mathfrak{x}$ such that $c_{v,w} \neq 0$. Hence $\mathfrak{y} = \mathfrak{y}(\{o,v,w\})$ must equal $\mathfrak{x}$. If $o \notin [v,w]$, then o is not an interior point of $\mathfrak{y}$, contradicting $o \in \mathrm{Int}(\mathfrak{x})$ and $\mathfrak{y} = \mathfrak{x}$. Hence $o \in [v,w]$, and so $\mathfrak{y} = \mathfrak{y}(\{v,w\})$.

Suppose first that $\mathfrak{x}$ is the subtree of Figure 1(b). In this case we now know that $\mathcal{H}'_{\mathfrak{x}}$ consists of functions

$$F = \sum_{\substack{i,j=0 \\ i\neq j}}^{q} c_{i,j}\mathbf{1}_{\Omega_o(a_i)} \otimes \mathbf{1}_{\Omega_o(a_j)}.$$

The coefficients $c_{i,j}$ must satisfy the conditions of (2.2). For if we fix $i \in \{0,\dots,q\}$ and let $\mathfrak{y}$ consist of o, a_i and the edge joining them, then we see that $K(\mathfrak{y})$ acts transitively on $\{a_j : j \neq i\}$. Fix some $j \neq i$, and let $(\omega_1,\omega_2) \in \Omega_o(a_i)\times\Omega_o(a_j)$. For each $r \neq i$ choose $k_r \in K(\mathfrak{y})$ such that $a_j = k_r a_r$. Then $0 = (P_{\mathfrak{y}}F)(\omega_1,\omega_2)$ is the average of the integrals over the q cosets $K'k_r$ of K' in $K(\mathfrak{y})$. Here $K' = \{k \in G : ko = o, ka_i = a_i \text{ and } ka_j = a_j\}$. As $F(g^{-1}\omega_1, g^{-1}\omega_2) = c_{i,r}$ for all $g \in K'k_r$, we see that the first of the conditions (2.2) holds. Similarly, fixing $(\omega_1,\omega_2) \in \Omega_o(a_j)\times\Omega_o(a_i)$, where $i \neq j$, we find that the second of the conditions (2.2) holds. Using $\xi_{a_i} = (q+1)\mathbf{1}_{\Omega_o(a_i)} - \mathbf{1}$, and (2.2), we find that

$$\sum_{\substack{i,j=0 \\ i\neq j}}^{q} c_{i,j}\mathbf{1}_{\Omega_o(a_i)} \otimes \mathbf{1}_{\Omega_o(a_j)} = \frac{1}{(q+1)^2}\sum_{\substack{i,j=0 \\ i\neq j}}^{q} c_{i,j}\xi_{a_i}\otimes\xi_{a_j},$$

and so we have shown that F can be written in the form (2.1), with the new $c_{i,j}$'s, namely the $c_{i,j}/(q+1)^2$'s satisfying (2.2). This completes the proof of part (i) of the proposition.

Now suppose that $\mathfrak{x}$ is not the subtree of Figure 1(b), nor in its G-orbit. We know that $\mathfrak{x} = \mathfrak{y}(\{v,w\})$ for some $v, w \in \partial\mathfrak{x}$ such that $o \in [v,w]$. Replacing $\mathfrak{x}$ by $g\mathfrak{x}$ for some $g \in G$, we may suppose that $\mathfrak{x}$ is the subtree of the type described in Figure 1(c_r). Let $F \in \mathcal{H}_\mathfrak{x}$. We know that F is of the form (2.12). Now $c_{v,w} = 0$ for any distinct $v, w \in \partial\mathfrak{x}$ unless $v = a_i$ and $w = y_j$ or vice versa for some $i, j \in \{1, \dots, q\}$. For otherwise o, v and w are contained in a complete subtree $\mathfrak{y} \subsetneqq \mathfrak{x}$, which implies that $c_{v,w} = 0$, as we saw above. So

$$F = \sum_{i,j=1}^{q} c_{i,j} \mathbf{1}_{\Omega_o(a_i)} \otimes \mathbf{1}_{\Omega_o(y_j)} + \sum_{i,j=1}^{q} d_{i,j} \mathbf{1}_{\Omega_o(y_j)} \otimes \mathbf{1}_{\Omega_o(a_i)} \tag{2.13}$$

for suitable constants $c_{i,j}$ and $d_{i,j}$. Fix i and j, let $(\omega_1, \omega_2) \in \Omega_o(a_i) \times \Omega_o(y_j)$, and let $\mathfrak{y}$ be the complete subtree obtained by deleting the y_k's and the edges $\{o, y_k\}$. Then $(P_\mathfrak{y} F)(\omega_1, \omega_2) = 0$ implies that $\sum_{r=1}^q c_{i,r} = 0$. Similarly, if we let $\mathfrak{y}$ be the complete subtree obtained by deleting the a_k's and the edges $\{o, a_k\}$, we obtain $\sum_{r=1}^q c_{r,j} = 0$. The same relations on the $d_{i,j}$'s are obtained by fixing $(\omega_1, \omega_2) \in \Omega_o(y_j) \times \Omega_o(a_i)$ and using the same $\mathfrak{y}$'s.

Now using these conditions on the $c_{i,j}$'s and $d_{i,j}$'s, together with $\xi_{a_i} = (q+1)\mathbf{1}_{\Omega_o(a_i)} - \mathbf{1}$ and $\xi_{y_j} = N_{r+1}\mathbf{1}_{\Omega_o(y_j)} - N_r \mathbf{1}_{\Omega_o(z)}$, we can write the sum in (2.13) in the form (2.3), where the (new) $c_{i,j}$'s and $d_{i,j}$'s satisfy (2.4). □

Lemma 2.2. *For each $\mathfrak{x}$ such that $|\mathfrak{x}| > 1$, $\mathcal{H}'_\mathfrak{x}$ is contained in H_2.*

Proof. Fix $\xi \in \mathcal{H}'_\mathfrak{x}$, and define $u : G \to \mathbb{C}$ by

$$u(g) = \langle \pi(g)\xi, \mathbf{1} \rangle_{s_1,s_2} .$$

Then $u(k_1 g k_2) = u(g)$ for all $g \in G$, $k_1 \in K$ and $k_2 \in K(\mathfrak{x})$. Also, if $\mathfrak{y}$ denotes a finite complete subtree of T, then

$$\int_{K(\mathfrak{y})} u(gk)\, dk = 0 \quad \text{for all } g \in G, \text{ if } \mathfrak{y} \subsetneqq \mathfrak{x}. \tag{2.14}$$

Suppose first that we are not in the special case. Then because u is a bi-$K(\mathfrak{x})$-invariant function satisfying (2.14), it must be supported in $\tilde{K}(\mathfrak{x})$ [2, Prop. III.3.2]. In Case (i) of Proposition 2.1 above, $\tilde{K}(\mathfrak{x}) = K$. As u is left K-invariant, it must be constant on its support. But $\mathfrak{y} = \{o\} \subsetneqq \mathfrak{x}$, and so

$$u(1) = \langle \xi, \mathbf{1} \rangle_{s_1,s_2} = \langle \xi, P_\mathfrak{y}\mathbf{1} \rangle_{s_1,s_2} = \langle P_\mathfrak{y}\xi, \mathbf{1} \rangle_{s_1,s_2} = \langle 0, \mathbf{1} \rangle_{s_1,s_2} = 0,$$

and so $u = 0$.

If we are in Case (ii) of Proposition 2.1, then u is supported on $\tilde{K}(\mathfrak{x}) = (K_o \cap K_z) \cup (K_o \cap K_z)g_0$, where $g_0 o = z$ and $g_0 z = o$, and where $K_x = \{g \in G : gx = x\}$ for any $x \in T$. As u is left K-invariant, it must be constant on $K_o = K$. But $K \not\subset \tilde{K}(\mathfrak{x})$, and so u is zero on K, and hence on $K_o \cap K_z$. Note that $\pi(g_0)\xi \in \mathcal{H}'_\mathfrak{x}$. The same reasoning applied to $v(g) = \langle \pi(g)(\pi(g_0)\xi), \mathbf{1} \rangle_{s_1,s_2}$ shows that $u(g_0) = v(1) = 0$. It follows that $u = 0$ again.

In the special case, u is not supported on $\tilde{K}(\mathfrak{x})$, but is determined by $\alpha = u(1)$ and $\beta = u(g_0)$, where $g_0 o = a$ and $g_0 a = o$. To see this, (cf. [2, Proposition III.2.3]) let $\tilde{\mathcal{E}}$ denote the set of directed edges of T, i.e., the set of ordered pairs (x, y), where

$x, y \in T$ and $d(x, y) = 1$. Defining $g.(x, y) = (gx, gy)$, we see that G acts transitively on $\tilde{\mathcal{E}}$. By right $K(\mathfrak{x})$-invariance of u, we can define $\tilde{u} : \tilde{\mathcal{E}} \to \mathbb{C}$ by

$$\tilde{u}((go, ga)) = u(g).$$

We know that for $\mathfrak{y} = \{o\}$ and $\{a\}$,

$$\int_{K(\mathfrak{y})} u(gk)\, dk = 0 \quad \text{for all } g \in G.$$

Thus for each vertex v, the sum of $\tilde{u}((x, y))$ over the edges (x, y) for which $x = v$ is zero, as is the sum over the edges (x, y) for which $y = v$. Using this and the left $K(\mathfrak{x})$-invariance of u, we see how u's values are determined: each directed edge (x, y) lies on a doubly infinite geodesic of one of the following two types, and the values of $\tilde{u}$ are as indicated:

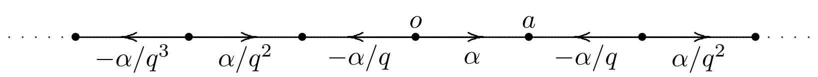


Figure 2(a).

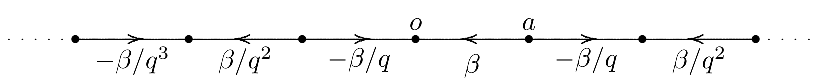


Figure 2(b).

Arguing exactly as in Case (ii), we see that $u(1) = 0 = u(g_0)$. Hence $\alpha = \beta = 0$, and so $u = 0$. □

Because $K(\mathfrak{x})$ is a normal subgroup of $\tilde{K}(\mathfrak{x})$, the group $\tilde{K}(\mathfrak{x})$ acts on the finite dimensional subspace $\mathcal{H}'_{\mathfrak{x}}$ of $\mathcal{H}_{s_1,s_2}$. Let τ denote the representation of $\tilde{K}(\mathfrak{x})$ thus obtained. By definition of $\mathcal{H}'_{\mathfrak{x}}$, τ is a sum of representations in $\big(\tilde{K}(\mathfrak{x})\big)^{\wedge}_0$. Let σ be the representation of G obtained by inducing τ from $\tilde{K}(\mathfrak{x})$ to G. Its representation space $\mathcal{H}_\sigma$ is the space of functions $f : G \to \mathcal{H}'_{\mathfrak{x}}$ such that $f(gk) = \tau(k^{-1})\big(f(g)\big)$ for all $g \in G$ and $k \in \tilde{K}(\mathfrak{x})$ and such that $\int_G \|f(g)\|^2\, dg < \infty$.

Lemma 2.3. *Assume that* $|\mathfrak{x}| > 2$. *Then* $\sigma = \mathrm{Ind}(\tau)$ *is equivalent to the sum of the discrete series subrepresentations of* π *having minimal tree* $\mathfrak{x}$.

Proof. Let $g_0\tilde{K}(\mathfrak{x}), g_1\tilde{K}(\mathfrak{x}), \dots$ be the distinct left cosets of $\tilde{K}(\mathfrak{x})$ in G. Notice that, if $v, v' \in \mathcal{H}'_{\mathfrak{x}}$, then

$$\langle \pi(g_i)v, \pi(g_j)v' \rangle_{s_1,s_2} = \delta_{i,j} \langle v, v' \rangle_{s_1,s_2} \tag{2.15}$$

For if we set $u(g) = \langle \pi(g)v, v' \rangle_{s_1,s_2}$, then u is bi-$K(\mathfrak{x})$-invariant, and satisfies (2.14) above. Hence u is supported on $\tilde{K}(\mathfrak{x})$ by [2, Prop. III.3.2], and so (2.15) holds. It follows that we may define $T : \mathcal{H}_\sigma \to \mathcal{H}_{s_1,s_2}$ by

$$Tf = \big(m\big(\tilde{K}(\mathfrak{x})\big)\big)^{1/2} \sum_i \pi(g_i)\big(f(g_i)\big).$$

It is clear from (2.15) that T is an isometry. It intertwines σ and π, for if $g \in G$, we can write $g^{-1}g_i = g_j k$ for some $k \in \tilde{K}(\mathfrak{x})$ and index j depending on g and i. Now

$$\big(\sigma(g)f\big)(g_i) = f(g^{-1}g_i) = f(g_j k) = \pi(k^{-1})\big(f(g_j)\big),$$

so that

$$\pi(g_i)\big(\big(\sigma(g)f\big)(g_i)\big) = \pi(g_i)\big(\pi(k^{-1})\big(f(g_j)\big)\big) = \pi(g)\big(\pi(g_j)\big(f(g_j)\big)\big).$$

As i varies over $0, 1, \ldots$, so does j, and so $T(\sigma(g)f) = \pi(g)\big(Tf\big)$.

The image of T is the sum of the discrete series subrepresentations of π having minimal tree $\mathfrak{x}$. For if σ_0 is such a subrepresentation, with representation space $\mathcal{H}_{\sigma_0} \subset \mathcal{H}_{s_1,s_2}$, then $\mathcal{H}_{\sigma_0}$ contains a nonzero $v_0 \in \mathcal{H}'_{\mathfrak{x}}$. Define $f \in \mathcal{H}_\sigma$ by $f(k) = \pi(k^{-1})v_0$ if $k \in \tilde{K}(\mathfrak{x})$ and $f(k) = 0$ otherwise. Then $T(f) = v_0$. As the image of T is closed and π invariant, it contains $\mathcal{H}_{\sigma_0}$. □

The last lemma reduces the problem of determining the decomposition (1.2) to (A): for $\mathfrak{x}$ as in cases (i) and (ii) of Proposition 2.1, determining the decomposition into irreducibles of the representation τ of $\tilde{K}(\mathfrak{x})$ defined before Lemma 2.3; and (B): determining which of the special representations of G occur in (1.2), and their multiplicities. These steps are carried out in the next three sections.

3. Case (i) of Proposition 2.1

In this case, $\mathcal{H}'_{\mathfrak{x}}$ is $q(q-1)-1$ dimensional, and consists of elements $\sum_{a,b\in\mathcal{C}_1} t_{a,b}\, \xi_a \otimes \xi_b$, where $\mathcal{C}_1 = \{x \in T : d(o,x) = 1\}$, and where the numbers $t_{a,b} \in \mathbb{C}$, $a, b \in \mathcal{C}_1$, satisfy the conditions

$$t_{a,a} = 0 \text{ for each } a \in \mathcal{C}_1,$$

$$\sum_{a\in\mathcal{C}_1} t_{a,b} = 0 \text{ for each } b \in \mathcal{C}_1, \quad \text{and} \quad \sum_{b\in\mathcal{C}_1} t_{a,b} = 0 \text{ for each } a \in \mathcal{C}_1. \tag{3.1}$$

Proposition 3.1. *The representation $g \mapsto \pi(g)_{|\mathcal{H}'_{\mathfrak{x}}}$ of $\tilde{K}(\mathfrak{x})$ on $\mathcal{H}'_{\mathfrak{x}}$ is (a) one dimensional if $q = 2$, and (b) the sum of an irreducible subrepresentation of dimension $(q+1)(q-2)/2$ and one of dimension $q(q-1)/2$ when $q > 2$. These subrepresentations are described below.*

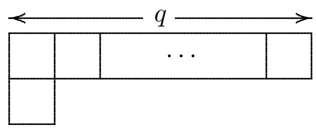
Proof. In this case $\tilde{K}(\mathfrak{x}) = K = \{g \in G : go = o\}$. Suppose that $g \in \tilde{K}(\mathfrak{x})$. Then

$$\pi(g)\Big(\sum_{a,b\in\mathcal{C}_1} t_{a,b}\xi_a \otimes \xi_b\Big) = \sum_{a,b\in\mathcal{C}_1} t_{a,b}\xi_{ga} \otimes \xi_{gb} = \sum_{a,b\in\mathcal{C}_1} t_{g^{-1}a,g^{-1}b}\, \xi_a \otimes \xi_b\,.$$

Now $\mathcal{H}'_{\mathfrak{x}}$ may be decomposed into a sum of two subspaces invariant under $\pi(g)$ for each $g \in \tilde{K}(\mathfrak{x})$. In fact, $\mathcal{H}'_{\mathfrak{x}} = V_+ \oplus V_-$, where V_+ (resp. V_-) is the subspace of $\mathcal{H}'_{\mathfrak{x}}$ consisting of elements $\sum_{a,b\in\mathcal{C}_1} t_{a,b}\xi_a \otimes \xi_b$ for which $t_{a,b} = t_{b,a}$ (resp. $t_{a,b} = -t_{b,a}$) for all $a, b \in \mathcal{C}_1$. It is easy to see that $\dim(V_+) = (q+1)(q-2)/2$ and that $\dim(V_-) = q(q-1)/2$ $(= \dim(V_+) + 1)$. Notice that $V_+ = \{0\}$ if $q = 2$.

Restriction to V_+ and V_- gives representations π_+ and π_- of $\tilde{K}(\mathfrak{x})$. It is well known that they are irreducible. For if we write $\mathcal{C}_1 = \{a_0, \ldots, a_q\}$, then for $g \in \tilde{K}(\mathfrak{x})$, we have $ga_i = a_{f(g)i}$ for some $f(g) \in S_{q+1}$. This defines a surjective homomorphism $f : \tilde{K}(\mathfrak{x}) \to S_{q+1}$ (with kernel $K(\mathfrak{x})$). Let $U = \{(x_j) \in \mathbb{C}^{q+1} : \sum_{j=0}^{q} x_j =$

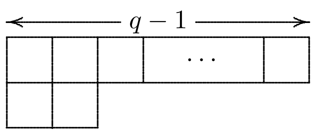
$0\}$, and let λ be the natural representation of S_{q+1} on U: $\lambda(\mu)(x_0,\dots,x_q) = (x_{\mu^{-1}0},\dots,x_{\mu^{-1}q})$. This is the irreducible representation of S_{q+1} corresponding to the partition $(q,1)$ of $q+1$ (see [3, Lemma 2.2.19(iii)]), i.e. to the Young diagram



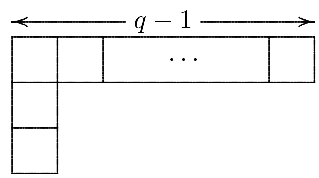
The representation π of $\tilde{K}(\mathfrak{r})$ on $\mathcal{H}'_{\mathfrak{r}}$ is equivalent to a subrepresentation of $(\lambda\otimes\lambda)\circ f$, an intertwining operator being

$$\sum_{i,j=0}^{q} t_{i,j}\,\xi_{a_i}\otimes\xi_{a_j}\mapsto\sum_{i,j=0}^{q} t_{i,j}\,\mathbf{e}_i\otimes\mathbf{e}_j\,,$$

where $\mathbf{e}_0,\dots,\mathbf{e}_q$ is the usual basis of $\mathbb{C}^{q+1}$. But the decomposition of $\lambda\otimes\lambda$ into irreducible subrepresentations is given on page 97 in [3] (where $n=q+1$). Two of these irreducible components are the representations $\pi_{q-1,2}$ and $\pi_{q-1,1,1}$ of S_{q+1} corresponding to the partitions $(q-1,2)$ and $(q-1,1,1)$ of $q+1$, i.e. to the Young diagrams



and



Comparing the dimensions of these two representations (see [3, Theorem 2.3.21]), we see that these match those of π_+ and π_-, respectively. The above intertwining operator therefore gives equivalences between π_+ and $(\pi_{q-1,2})\circ f$ and between π_- and $(\pi_{q-1,1,1})\circ f$. When $q=2$, it is easy to verify that the representation $g\mapsto\pi(g)_{|\mathcal{H}'_{\mathfrak{r}}}$ of $\tilde{K}(\mathfrak{r})$ on $\mathcal{H}'_{\mathfrak{r}}$ is (equivalent to) $\epsilon\circ f$, where ϵ is the sign character of S_3. $\square$

4. Case (ii) of Proposition 2.1

Referring to Figure 1(c_r), let A denote the set of neighbors of o other than z_1, and let Y denote the neighbors of z other than z_{r-1}. Then $\mathcal{H}'_{\mathfrak{r}}$ consists of the elements

$$\sum_{a\in A,y\in Y} s_{a,y}\,\xi_a\otimes\xi_y+\sum_{a\in A,y\in Y} t_{a,y}\,\xi_y\otimes\xi_a,$$

where

$$\begin{aligned}&\sum_{a\in A} s_{a,y}=\sum_{a\in A} t_{a,y}=0 \quad\text{for each } y\in Y,\text{ and}\\ &\sum_{y\in Y} s_{a,y}=\sum_{y\in Y} t_{a,y}=0 \quad\text{for each } a\in A.\end{aligned}\tag{4.1}$$

Proposition 4.1. *The representation $g\mapsto\pi(g)_{|\mathcal{H}'_{\mathfrak{r}}}$ of $\tilde{K}(\mathfrak{r})$ on $\mathcal{H}'_{\mathfrak{r}}$ is the sum of two inequivalent irreducible subrepresentations, each of dimension $(q-1)^2$. The two subrepresentations are described below.*

Proof. Suppose that $g \in \tilde{K}(\mathfrak{x})$. Then either (i): $go = o$ and $gz = z$, in which case g permutes A and Y, and

$$\begin{aligned}\pi(g)\Big(\sum_{a\in A,y\in Y} c_{a,y}\xi_a\otimes\xi_y + \sum_{a\in A,y\in Y} d_{a,y}\xi_y\otimes\xi_a\Big)\\ = \sum_{a\in A,y\in Y} c_{a,y}\xi_{ga}\otimes\xi_{gy} + \sum_{a\in A,y\in Y} d_{a,y}\xi_{gy}\otimes\xi_{ga}\\ = \sum_{a\in A,y\in Y} c_{g^{-1}a,g^{-1}y}\,\xi_a\otimes\xi_y + \sum_{a\in A,y\in Y} d_{g^{-1}a,g^{-1}y}\,\xi_y\otimes\xi_a\,,\end{aligned}$$

or (ii): $go = z$ and $gz = o$, in which case g interchanges A and Y, and we find using (4.1) that

$$\begin{aligned}&\pi(g)\Big(\sum_{a\in A,y\in Y} c_{a,y}\xi_a\otimes\xi_y + \sum_{a\in A,y\in Y} d_{a,y}\xi_y\otimes\xi_a\Big)\\ &= \Big(\frac{s_2}{s_1}\Big)^r\sum_{a\in A,y\in Y} c_{a,y}\xi_{ga}\otimes\xi_{gy} + \Big(\frac{s_1}{s_2}\Big)^r\sum_{a\in A,y\in Y} d_{a,y}\xi_{gy}\otimes\xi_{ga}\\ &= \Big(\frac{s_1}{s_2}\Big)^r\sum_{a\in A,y\in Y} d_{g^{-1}y,g^{-1}a}\,\xi_a\otimes\xi_y + \Big(\frac{s_2}{s_1}\Big)^r\sum_{a\in A,y\in Y} c_{g^{-1}y,g^{-1}a}\,\xi_y\otimes\xi_a.\end{aligned}$$

Let $\epsilon : \tilde{K}(\mathfrak{x}) \to \{-1, 1\}$ be the character defined by setting $\epsilon(g) = +1$ in case (i) and -1 in case (ii). Now $\mathcal{H}'_{\mathfrak{x}}$ may be decomposed into a sum of two subspaces invariant under $\pi(g)$ for each $g \in \tilde{K}(\mathfrak{x})$. In fact, $\mathcal{H}'_{\mathfrak{x}} = V_+ \oplus V_-$, where for $\delta = \pm1$, V_δ is the subspace of $\mathcal{H}'_{\mathfrak{x}}$ consisting of elements of the form

$$\sum_{a\in A,y\in Y} c_{a,y}\xi_a\otimes\xi_y + \delta\Big(\frac{s_2}{s_1}\Big)^r\sum_{a\in A,y\in Y} c_{a,y}\xi_y\otimes\xi_a\,.$$

Restriction to V_+ and V_- gives representations π_+ and π_- of $\tilde{K}(\mathfrak{x})$. Clearly π_- is equivalent to the product of π_+ by ϵ.

The group S_2 of permutations of $\{1, 2\}$ acts on the product $S_q \times S_q$ of 2 copies of S_q: for $p \in S_2$ and $\mu_1, \mu_2 \in S_q$ we set

$$p\cdot(\mu_1,\mu_2) = (\mu_{p^{-1}1},\mu_{p^{-1}2}).$$

We can therefore form the semidirect product $(S_q \times S_q) \rtimes S_2$ (the *wreath product* S_q wr S_2 of S_q by S_2 — see [3, p. 132]), whose elements are triples (μ_1, μ_2, p), where $p \in S_2$ and $\mu_1, \mu_2 \in S_q$, and where multiplication is defined by

$$(\mu_1,\mu_2,p)(\mu'_1,\mu'_2,p') = (\mu_1\mu'_{p^{-1}1},\mu_2\mu'_{p^{-1}2},pp').$$

There is a group homomorphism $f : \tilde{K}(\mathfrak{x}) \to (S_q\times S_q)\rtimes S_2$, defined as follows. Write $A = \{a_1, \ldots, a_q\}$ and $Y = \{y_1, \ldots, y_q\}$. For g as in case (i), we have $ga_i = a_{\varphi(g)i}$ and $gy_i = y_{\vartheta(g)i}$ for some $\varphi(g), \vartheta(g) \in S_q$, and we define $f(g)$ to be $(\varphi(g), \vartheta(g), 1)$; for g as in case (ii), $ga_i = y_{\varphi'(g)i}$ and $gy_i = a_{\vartheta'(g)i}$ for some $\varphi'(g), \vartheta'(g) \in S_q$, and we set $f(g) = (\vartheta'(g), \varphi'(g), (1\,2))$. It is easy to check that f is a homomorphism, which is clearly surjective.

Let $U = \{(x_j) \in \mathbb{C}^q : \sum_{j=1}^q x_j = 0\}$, and let λ be the natural representation of S_q on U: $\lambda(\mu)(x_1, \ldots, x_q) = (x_{\mu^{-1}1}, \ldots, x_{\mu^{-1}q})$. This is the irreducible representation of S_q corresponding to the partition $(q - 1, 1)$ of q (cf. Section 3 above). There is a representation $\tilde\lambda$ of $(S_q \times S_q) \rtimes S_2$ on $U \otimes U$ determined by

$$\tilde\lambda(\mu_1,\mu_2,p)\big(u_1 \otimes u_2\big) = \big(\lambda(\mu_1)(u_{p^{-1}1})\big) \otimes \big(\lambda(\mu_2)(u_{p^{-1}2})\big)$$

(see [3, p. 147]). This is clearly irreducible, because its restriction to the index 2 subgroup $\{(\mu_1,\mu_2,1) : \mu_1,\mu_2 \in S_q\} \cong S_q \times S_q$ of $(S_q \times S_q) \rtimes S_2$ is already irreducible, being equivalent to the outer tensor product of two copies of λ on $S_q \times S_q$.

Finally, the representations π_+ and $\tilde\lambda \circ f$ of $\tilde K(\mathfrak{r})$ are equivalent, an intertwining operator being

$$\sum_{i,j=1}^{q} c_{i,j}\Big(\xi_{a_i} \otimes \xi_{y_j} + \Big(\frac{s_2}{s_1}\Big)^r \xi_{y_j} \otimes \xi_{a_i}\Big) \mapsto \sum_{i,j=1}^{q} c_{i,j}\mathbf{e}_i \otimes \mathbf{e}_j\,,$$

where $\mathbf{e}_1,\dots,\mathbf{e}_q$ is the usual basis of $\mathbb{C}^q$. □

5. The special case

Let $\mathfrak{r}$ be as in Figure 1(a). We must first describe $\mathcal{H}'_{\mathfrak{r}}$. For $n \geq 1$, let

$$X_n = \{(\omega_1,\omega_2) \in \Omega_o(a) \times \Omega_o(a) : \omega_1 \neq \omega_2 \text{ and } d(o,\omega_1 \wedge \omega_2) = n\}$$

($= D_{a,n}$, see (2.5)). Let $\Omega'_o(x)$ denote $\Omega \setminus \Omega_o(x)$, and for $n \geq 0$ let

$$Y_n = \{(\omega_1,\omega_2) \in \Omega'_o(a) \times \Omega'_o(a) : \omega_1 \neq \omega_2 \text{ and } d(o,\omega_1 \wedge \omega_2) = n\}.$$

Also, let

$$\Delta_a = \{(\omega_1,\omega_2) \in \Omega_o(a) \times \Omega_o(a) : \omega_1 = \omega_2\}$$

and

$$\Delta'_a = \{(\omega_1,\omega_2) \in \Omega'_o(a) \times \Omega'_o(a) : \omega_1 = \omega_2\}.$$

Let $Z = \Omega_o(a) \times \Omega'_o(a)$ and $Z' = \Omega'_o(a) \times \Omega_o(a)$. Then

$$\Omega \times \Omega = \Delta_a \cup \Delta'_a \cup \bigcup_{n\geq 1} X_n \cup \bigcup_{n \geq 0} Y_n \cup Z \cup Z',$$

a disjoint union. We also need to define

$$X'_M = \{(\omega_1,\omega_2) \in \Omega_o(a) \times \Omega_o(a) : \omega_1 = \omega_2, \text{ or } \omega_1 \neq \omega_2 \text{ and } d(o,\omega_1 \wedge \omega_2) > M\}.$$

for $M \geq 1$, and

$$Y'_M = \{(\omega_1,\omega_2) \in \Omega'_o(a) \times \Omega'_o(a) : \omega_1 = \omega_2, \text{ or } \omega_1 \neq \omega_2 \text{ and } d(o,\omega_1 \wedge \omega_2) > M\}.$$

If $F \in \mathcal{K}(\Omega \times \Omega)$ satisfies $P_{\mathfrak{r}}F = F$, then as in the case $|\mathfrak{r}| > 2$, we can write

$$F = \sum_{n=1}^{M} x_n \mathbf{1}_{X_n} + x\mathbf{1}_{X'_M} + \sum_{n=0}^{M} y_n \mathbf{1}_{Y_n} + y\mathbf{1}_{Y'_M} + z\mathbf{1}_Z + z'\mathbf{1}_{Z'},$$

for some integer $M \geq 1$ and constants x_n, x, y_n, y, z, z'.

Lemma 5.1. *For f equal to any of $\mathbf{1}_{X_n}$, $\mathbf{1}_{Y_n}$, $\mathbf{1}_{X'_n}$ or $\mathbf{1}_{Y'_n}$, the following estimates hold for the norm $\|f\|_{s_1,s_2}$ of f in $\mathcal{H}_{s_1,s_2}$.*

$$\|f\|_{s_1,s_2} = \begin{cases} O\Big(n^{\frac12}\max\Big\{\frac{1}{q^n}, \Big(\frac{|s_1 s_2|}{\sqrt{q}}\Big)^n\Big\}\Big) & \text{if } s_1^2, s_2^2 \in (1/q,q), \\ O\Big(\Big(\frac{|s_2|}{\sqrt{q}}\Big)^n\Big) & \text{if } |s_1| = 1 \text{ and } s_2^2 \in (1/q,q), \\ O\Big(\Big(\frac{|s_1|}{\sqrt{q}}\Big)^n\Big) & \text{if } s_1^2 \in (1/q,q) \text{ and } |s_2| = 1, \\ O\Big(\frac{1}{q^{n/2}}\Big) & \text{if } |s_1|, |s_2| = 1. \end{cases}$$

Proof. Let us write $\tilde{J}$ in place of $\tilde{J}_{s_1,s_2}$. Recall that if $v \in T$ and $n, M \geq |v| = d(o,v)$, then $D_{v,n}$ and $D'_{v,M}$ are defined as in (2.5) and (2.8). Let $b \in T$ satisfy $|b| = 1$. Then X_n is $D_{a,n}$, and Y_n is the union of the sets $D_{b,n}$, where $|b| = 1$ and $b \neq a$. Similarly, X'_n is $D'_{a,n}$, and Y'_n is the union of the sets $D'_{b,n}$, where $|b| = 1$ and $b \neq a$. So it is enough to fix b with $|b| = 1$, and estimate $\langle f_n, f_n\rangle_{s_1,s_2} = \langle f_n, \tilde{J} f_n\rangle'$ and $\langle f'_n, f'_n\rangle_{s_1,s_2} = \langle f'_n, \tilde{J} f'_n\rangle'$, where

$$f_n = \mathbf{1}_{D_{b,n}} \quad \text{and} \quad f'_n = \mathbf{1}_{D'_{b,n}}.$$

Moreover, since $f_n = f'_{n-1} - f'_n$, it is sufficient to estimate $\langle f'_n, \tilde{J} f'_n\rangle'$

To do this, let $n \geq 1$, let $|x| = n$, and consider the following element of $\mathcal{K}(\Omega\times\Omega)$:

$$\Sigma_x = \sum_{\substack{y:|y|=n+1,\\ y'=x}} \xi_y \otimes \xi_y. \tag{5.1}$$

Clearly Σ_x is supported on $\Omega_o(x) \times \Omega_o(x)$. Let us fix $\omega_1, \omega_2 \in \Omega_o(x)$, and evaluate both sides of (5.1) at (ω_1, ω_2). Let z_1 and z_2 denote the $(n+1)$th vertices on $[o, \omega_1)$ and $[o, \omega_2)$, respectively. If $z_1 \neq z_2$ (which just means that $(\omega_1, \omega_2) \in D_{x,n}$), then $\xi_y(\omega_1)\xi_y(\omega_2) = -N_n(N_{n+1} - N_n)$ if $y = z_1$ or z_2, and $\xi_y(\omega_1)\xi_y(\omega_2) = (-N_n)^2$ for the remaining $(q-2)$ y's satisfying $|y| = n+1$ and $y' = x$. Hence

$$\Sigma_x(\omega_1, \omega_2) = (q-2)N_n^2 - 2N_n(N_{n+1} - N_n) = Aq^{2n}$$

for $A = -(q+1)^2/q$. If $z_1 = z_2$ (which just means that $(\omega_1, \omega_2) \in D'_{x,n}$), then $\xi_y(\omega_1)\xi_y(\omega_2) = (N_{n+1} - N_n)^2$ if $y = z_1 = z_2$, and $\xi_y(\omega_1)\xi_y(\omega_2) = (-N_n)^2$ for the remaining $(q-1)$ y's satisfying $|y| = n+1$ and $y' = x$. Hence

$$\Sigma_x(\omega_1, \omega_2) = (q-1)N_n^2 + (N_{n+1} - N_n)^2 = Bq^{2n}$$

for $B = (q-1)(q+1)^2/q = -(q-1)A$.

Hence

$$\Sigma_x = Aq^{2n}\mathbf{1}_{D_{x,n}} + Bq^{2n}\mathbf{1}_{D'_{x,n}}.$$

If we now fix $b \in T$ satisfying $|b| = 1$, and sum this last identity over all x such that $b \in [o,x]$ and $|x| = n$, we obtain

$$\sum_{\substack{x:b\in[o,x],\\ |x|=n}} \Sigma_x = Aq^{2n} f_n + Bq^{2n} f'_n.$$

Now apply $\tilde{J}$ to both sides. Write $\tilde{\jmath}_n(s)$ to denote either 1 if $|s| = 1$, or $\jmath_n(s)$ if $s^2 \in (1/q, q)$. Using (1.3), we get

$$\begin{aligned}
\tilde{\jmath}_{n+1}(s_1)\tilde{\jmath}_{n+1}(s_2)\big(Aq^{2n} f_n + Bq^{2n} f'_n\big) &= \tilde{\jmath}_{n+1}(s_1)\tilde{\jmath}_{n+1}(s_2)\Big(\sum_{\substack{x:b\in[o,x],\\ |x|=n}} \Sigma_x\Big) \\
&= \tilde{J}\Big(\sum_{\substack{x:b\in[o,x],\\ |x|=n}} \Sigma_x\Big) \\
&= Aq^{2n}\tilde{J} f_n + Bq^{2n}\tilde{J} f'_n.
\end{aligned}$$

Cancelling Aq^{2n}, using $B = -(q-1)A$, and writing $\jmath^*_{n+1}$ in place of $\tilde{\jmath}_{n+1}(s_1)\tilde{\jmath}_{n+1}(s_2)$, we get

$$\tilde{J} f_n - (q-1)\tilde{J} f'_n = \jmath^*_{n+1}\big(f_n - (q-1) f'_n\big) \quad \text{if } n \geq 1. \tag{5.2}$$

If $n > 1$, we also have $f_n + f'_n = f'_{n-1}$, and hence

$$\tilde{J} f_n + \tilde{J} f'_n = \tilde{J} f'_{n-1} \quad \text{if } n > 1. \tag{5.3}$$

It is clear from (5.2) and (5.3) that for all $n \geq 1$,

$$\tilde{J} f'_n = \frac{1}{q^{n-1}} \tilde{J} f'_1 + \sum_{k=1}^{n} \beta'_{n,k} f_k + C'_n f'_n \tag{5.4}$$

for certain numbers $\beta'_{n,k}, C'_n$. Indeed, if we set $\beta'_{1,1} = C'_1 = 0$, then (5.4) holds for $n = 1$. Assume that $n > 1$, and that (5.4) holds for $n-1$ in place of n. If we subtract (5.2) from (5.3), we obtain

$$q\tilde{J} f'_n = \tilde{J} f'_{n-1} - \jmath^*_{n+1} f_n + (q-1)\jmath^*_{n+1} f'_n,$$

and so

$$\tilde{J} f'_n = \frac{1}{q}\Big(\frac{1}{q^{n-2}} \tilde{J} f'_1 + \sum_{k=1}^{n-1} \beta'_{n-1,k} f_k + C'_{n-1} f'_{n-1}\Big) - \frac{\jmath^*_{n+1}}{q} f_n + \frac{(q-1)\jmath^*_{n+1}}{q} f'_n,$$

which gives formula (5.4) if we set

$$\begin{aligned} \beta'_{n,k} &= \frac{1}{q}\beta'_{n-1,k} \quad \text{if } 1 \leq k \leq n-1, \\ \beta'_{n,n} &= \frac{1}{q}\Big(C'_{n-1} - \jmath^*_{n+1}\Big), \\ C'_n &= \frac{1}{q}\Big(C'_{n-1} + (q-1)\jmath^*_{n+1}\Big). \end{aligned} \tag{5.5}$$

We have used $f'_{n-1} = f_n + f'_n$ again here.

Now

$$\begin{aligned} \langle f'_n, f'_n \rangle_{s_1,s_2} &= \langle f'_n, \tilde{J} f'_n \rangle' \\ &= \langle f'_n, \frac{1}{q^{n-1}} \tilde{J} f'_1 + \sum_{k=1}^{n} \beta'_{n,k} f_k + C'_n f'_n \rangle' \\ &= \frac{1}{q^{n-1}} \langle f'_n, \tilde{J} f'_1 \rangle' + C'_n \langle f'_n, f'_n \rangle'. \end{aligned}$$

Now

$$\langle f'_n, f'_n \rangle' = (\nu_o \times \nu_o)(D'_{b,n}) = \frac{1}{(q+1)^2 q^n} = O\Big(\frac{1}{q^n}\Big).$$

Also,

$$\langle f'_n, \tilde{J} f'_1 \rangle' = O\Big((\nu_o \times \nu_o)(D'_{b,n})\Big) = O\Big(\frac{1}{q^n}\Big)$$

because $\tilde{J} f'_1$ is a bounded function. Hence

$$\langle f'_n, f'_n \rangle_{s_1,s_2} = \langle f'_n, \tilde{J} f'_n \rangle' = O\Big(\frac{1}{q^{2n}}\Big) + O\Big(\frac{C'_n}{q^n}\Big). \tag{5.6}$$

So we need only estimate C'_n. Using the third equation in (5.5), we see by induction that if $n \geq 2$, then

$$C'_n = \frac{q-1}{q^{n+2}}\Big(\jmath^*_3 q^3 + \cdots + \jmath^*_{n+1} q^{n+1}\Big). \tag{5.7}$$

To sum the finite series (5.7), we must now consider the various cases:

When $|s_1| = |s_2| = 1$, Then $\jmath_k^* = 1$ for each k, and we get $C_n' = 1 - 1/q^{n-1}$.

When $|s_1| = 1$ and $s_2^2 \in (1/q, q)$, (5.7) gives

$$C_n' = \frac{D_1}{q^n} + D_2 s_2^{2n} = O\Big(|s_2|^{2n}\Big),$$

for certain constants D_1 and D_2. When $s_1^2 \in (1/q, q)$ and $|s_2| = 1$, (5.7) similarly gives $C_n' = D_1/q^n + D_2 s_1^{2n}$ for certain constants D_1 and D_2.

Finally, when $s_1^2, s_2^2 \in (1/q, q)$, we get

$$C_n' = \frac{D_1}{q^n} + D_2(s_1 s_2)^{2n} \quad \text{if } (s_1 s_2)^2 q \neq 1,$$

for certain constants D_1 and D_2. Also,

$$C_n' = \frac{D_1^*(n-1)}{q^n} \quad \text{if } (s_1 s_2)^2 q = 1,$$

for some constant D_1^*.

With these formulas for C_n', Lemma 5.1 is proved. □

Lemma 5.2. *Suppose that $\xi \in \mathcal{H}_{\mathfrak{x}}'$. Form*

$$\begin{aligned} x_n &= \langle \xi, \mathbf{1}_{X_n}\rangle'/(\nu_o \times \nu_o)(X_n) \quad \text{if } n \geq 1, \quad z = \langle \xi, \mathbf{1}_Z\rangle'/(\nu_o \times \nu_o)(Z), \\ y_n &= \langle \xi, \mathbf{1}_{Y_n}\rangle'/(\nu_o \times \nu_o)(Y_n) \quad \text{if } n \geq 0, \quad z' = \langle \xi, \mathbf{1}_{Z'}\rangle'/(\nu_o \times \nu_o)(Z'). \end{aligned} \tag{5.8}$$

Then the series

$$\xi = \sum_{k=1}^{\infty} x_k \mathbf{1}_{X_k} + \sum_{k=0}^{\infty} y_k \mathbf{1}_{Y_k} + z\mathbf{1}_Z + z'\mathbf{1}_{Z'}\ . \tag{5.9}$$

converges in $\mathcal{H}_{s_1,s_2}$ to ξ. Moreover,

$$\begin{aligned} y_{2n} &= \tfrac{-1}{(q-1)(s_1 s_2)^{2n}}\big(z + z'\big), && \text{if } n \geq 0, \\ y_{2n+1} &= \tfrac{1}{(q-1)(s_1 s_2)^{2n}}\Big(\tfrac{z}{s_2^2} + \tfrac{z'}{s_1^2}\Big), && \text{if } n \geq 0, \\ x_{2n} &= -q y_{2n} && \text{if } n \geq 1, \\ x_{2n+1} &= -q y_{2n+1} && \text{if } n \geq 0, \end{aligned} \tag{5.10}$$

and so ξ is determined by z and z'. Thus $\mathcal{H}_{\mathfrak{x}}'$ is two dimensional.

Proof. If $\mathfrak{y} = \{o\}$, then it is easy to see that $P_{\mathfrak{y}}(\mathbf{1}_{X_n}) = (\mathbf{1}_{X_n} + \mathbf{1}_{Y_n})/(q+1)$ and $P_{\mathfrak{y}}(\mathbf{1}_{Y_n}) = q(\mathbf{1}_{X_n} + \mathbf{1}_{Y_n})/(q+1)$ for $n \geq 1$. If $\xi \in \mathcal{H}_{\mathfrak{x}}'$, then $P_{\mathfrak{y}}\xi = 0$ therefore implies that $x_n + q y_n = 0$ for $n = 1, 2, \ldots$, because $(\nu_o \times \nu_o)(Y_n) = q(\nu_o \times \nu_o)(X_n)$. Also, using $P_{\mathfrak{y}}\mathbf{1}_{Y_0} = ((q-1)/(q+1))(\mathbf{1}_{Y_0} + \mathbf{1}_Z + \mathbf{1}_{Z'})$, plus $(\nu_o \times \nu_o)(Y_0) = q(q-1)/(q+1)^2 = (q-1)((\nu_o \times \nu_o)(Z)) = (q-1)((\nu_o \times \nu_o)(Z'))$, we find that $(q-1)y_0 + z + z' = 0$.

If $\mathfrak{y} = \{a\}$, then we can calculate

$$\begin{aligned} P_{\mathfrak{y}}\mathbf{1}_{X_n} &= \tfrac{q}{q+1}\Big\{\mathbf{1}_{X_n} + \Big(\tfrac{s_1 s_2}{q}\Big)^2 \mathbf{1}_{Y_{n-2}}\Big\} && \text{if } n \geq 2, \\ P_{\mathfrak{y}}\mathbf{1}_{X_1} &= \tfrac{q-1}{q+1}\Big\{\mathbf{1}_{X_1} + \Big(\tfrac{s_2^2}{q}\Big)\mathbf{1}_Z + \Big(\tfrac{s_1^2}{q}\Big)\mathbf{1}_{Z'}\Big\}, \\ P_{\mathfrak{y}}\mathbf{1}_{Y_n} &= \tfrac{1}{q+1}\Big\{\mathbf{1}_{Y_n} + \Big(\tfrac{q}{s_1 s_2}\Big)^2 \mathbf{1}_{X_{n+2}}\Big\} && \text{if } n \geq 0, \\ P_{\mathfrak{y}}\mathbf{1}_Z &= \tfrac{1}{q+1}\Big\{\Big(\tfrac{q}{s_2^2}\Big)\mathbf{1}_{X_1} + \mathbf{1}_Z + \Big(\tfrac{s_1}{s_2}\Big)^2 \mathbf{1}_{Z'}\Big\}, && \text{and} \\ P_{\mathfrak{y}}\mathbf{1}_{Z'} &= \tfrac{1}{q+1}\Big\{\Big(\tfrac{q}{s_1^2}\Big)\mathbf{1}_{X_1} + \mathbf{1}_{Z'} + \Big(\tfrac{s_2}{s_1}\Big)^2 \mathbf{1}_Z\Big\}. \end{aligned} \tag{5.11}$$

For example, to see the first of these, consider $(\omega_1, \omega_2) \in \Omega \times \Omega$, $g \in K(\mathfrak{y})$, and ask when $(g^{-1}\omega_1, g^{-1}\omega_2) \in X_n$. If $(\omega_1, \omega_2) \in X_n$, there is a neighbor b of a other than o such that $\omega_1, \omega_2 \in \Omega_o(b)$. We find that $(g^{-1}\omega_1, g^{-1}\omega_2) \in X_n$ and $\delta(o, go, \omega_1) = \delta(o, go, \omega_2) = 0$ unless $go = b$. If $(\omega_1, \omega_2) \in Y_{n-2}$, then $(g^{-1}\omega_1, g^{-1}\omega_2) \in X_n$ and $\delta(o, go, \omega_1) = \delta(o, go, \omega_2) = -2$ unless $go = o$. Also, $(g^{-1}\omega_1, g^{-1}\omega_2) \in X_n$ for no other $(\omega_1, \omega_2) \in \Omega \times \Omega$ and $g \in K(\mathfrak{y})$.

As $\xi \in \mathcal{H}'_{\mathfrak{r}}$, $P_{\mathfrak{y}}\xi = 0$ implies that

$$0 = \langle P_{\mathfrak{y}}\xi, \mathbf{1}_{X_n}\rangle' = \langle \xi, P_{\mathfrak{y}}\tilde{J}^{-1}_{s_1,s_2}\mathbf{1}_{X_n}\rangle_{s_1,s_2}. \tag{5.12}$$

Because $o \notin \mathfrak{y}$, $P_{\mathfrak{y}}$ and $\tilde{J}^{-1}_{s_1,s_2}$ no longer commute, but the first equation in (2.11) is still valid, and therefore $P_{\mathfrak{y}}\tilde{J}^{-1}_{s_1,s_2} = \tilde{J}^{-1}_{s_1,s_2}\tilde{P}_{\mathfrak{y}}$, where $\tilde{P}_{\mathfrak{y}}$ is defined as was $P_{\mathfrak{y}}$, but with (s_1, s_2) replaced by $(\tilde{s}_1, \tilde{s}_2)$. Using the first of the equations (5.11), with (s_1, s_2) replaced by $(\tilde{s}_1, \tilde{s}_2)$, together with $(\nu_o \times \nu_o)(Y_{n-2}) = q^3(\nu_o \times \nu_o)(X_n)$, we obtain from (5.12)

$$0 = x_n + \frac{q}{(s_1 s_2)^2} y_{n-2} \quad \text{if } n \geq 2.$$

The same equations are obtained using the third equation in (5.11). Using any of the other three equations in (5.11), we also obtain

$$0 = (q-1)x_1 + \frac{q}{s_2^2}z + \frac{q}{s_1^2}z'.$$

It is now elementary to obtain the formulas (5.10) for the x_n's and y_n's.

Using the estimates of Lemma 5.1 and the estimates $x_k, y_k = O(1/|s_1 s_2|^k)$ which follow from (5.10), we see that the series converges in $\mathcal{H}_{s_1,s_2}$. Let $\xi^* \in \mathcal{H}_{s_1,s_2}$ denote the sum. To see that $\xi^* = \xi$, it is enough to check that $\langle \xi^*, f\rangle' = \langle \xi, f\rangle'$ for each $f = \mathbf{1}_{X_n}, \mathbf{1}_{X'_n}$, $n \geq 1$, for $f = \mathbf{1}_{Y_n}, \mathbf{1}_{Y'_n}$, $n \geq 0$, and for $f = \mathbf{1}_Z$ and $\mathbf{1}_{Z'}$. By (5.8), we need only check the cases $f = \mathbf{1}_{X'_n}$, $n \geq 1$, and $f = \mathbf{1}_{Y'_n}$, $n \geq 0$. Now

$$\mathbf{1}_{\Omega_o(a)\times\Omega_o(a)} = \mathbf{1}_{X_1} + \cdots + \mathbf{1}_{X_n} + \mathbf{1}_{X'_n},$$

and so to check that $\langle \xi^*, \mathbf{1}_{X'_n}\rangle' = \langle \xi, \mathbf{1}_{X'_n}\rangle'$, we need only show that $\langle \xi^*, \mathbf{1}_{X'_n}\rangle'$, $\langle \xi, \mathbf{1}_{X'_n}\rangle' \to 0$. To do this, it is enough to show that $\|\tilde{J}^{-1}_{s_1,s_2}\mathbf{1}_{X'_n}\|_{s_1,s_2} \to 0$ as $n \to \infty$. But

$$\begin{aligned}
\|\tilde{J}^{-1}_{s_1,s_2}\mathbf{1}_{X'_n}\|^2_{s_1,s_2} &= \langle \tilde{J}^{-1}_{s_1,s_2}\mathbf{1}_{X'_n}, \tilde{J}^{-1}_{s_1,s_2}\mathbf{1}_{X'_n}\rangle_{s_1,s_2} \\
&= \langle \tilde{J}^{-1}_{s_1,s_2}\mathbf{1}_{X'_n}, \mathbf{1}_{X'_n}\rangle' \\
&= \|\mathbf{1}_{X'_n}\|^2_{s_1^{-1},s_2^{-1}} \\
&\to 0 \quad \text{by Lemma 5.1, with } (s_1, s_2) \text{ replaced by } (s_1^{-1}, s_2^{-1}).
\end{aligned}$$

The corresponding facts for Y'_n in place of X'_n are proved in the same way. □

Proposition 5.3. *Let $\mathcal{H}_{\mathrm{sp}}$ denote the closed linear span of $\{\pi(g)\xi : \xi \in \mathcal{H}'_{\mathfrak{r}}\}$. Then $\mathcal{H}_{\mathrm{sp}}$ is the the sum $\mathcal{H}_+ \oplus \mathcal{H}_-$ of two invariant subspaces. The restrictions of π to $\mathcal{H}_+$ and $\mathcal{H}_-$ are equivalent to the two distinct special representations of G.*

Proof. Let $g_0 \in G$ satisfy $g_0 o = a$, $g_0 a = o$ and $g_0^2 = 1$. Let us find $z, z' \in \mathbb{C}$ such that the corresponding element ξ^+ of $\mathcal{H}'_{\mathfrak{r}}$ satisfies $\pi(g_0)\xi^+ = \xi^+$. It is easy to calculate that

$$\begin{aligned}\pi(g_0)\mathbf{1}_{X_k} &= \frac{s_1s_2}{q}\mathbf{1}_{Y_{k-1}} \quad \text{for } k \geq 1,\\ \pi(g_0)\mathbf{1}_{Y_k} &= \frac{q}{s_1s_2}\mathbf{1}_{X_{k+1}} \quad \text{for } k \geq 0,\\ \pi(g_0)\mathbf{1}_{Z} &= \frac{s_1}{s_2}\mathbf{1}_{Z'},\\ \pi(g_0)\mathbf{1}_{Z'} &= \frac{s_2}{s_1}\mathbf{1}_{Z}.\end{aligned}$$

Using these, we see that the condition $\pi(g_0)\xi^+ = \xi^+$ amounts to the conditions

$$y_k = \frac{s_1s_2}{q}x_{k+1} \quad \text{for } k \geq 0, \quad \text{and} \quad z' = \frac{s_1}{s_2}z,$$

and using (5.10), we see that the second of these conditions implies the first. So if we set $z = s_2$, $z' = s_1$, and substitute these values of z, z' into the formulas (5.10) for the x_k's and y_k's, we get an element $\xi^+ \in \mathcal{H}'_{\mathfrak{r}}$ such that $\pi(g_0)\xi^+ = \xi^+$. In fact, $\pi(g)\xi^+ = \xi^+$ for all $g \in \tilde{K}(\mathfrak{r})$. Notice that

$$y_n = \frac{(-1)^{n-1}(s_1+s_2)}{(q-1)(s_1s_2)^n} \quad \text{for } n \geq 0,$$

and so in the case $s_2 = -s_1$, $\xi^+ = s_1(\mathbf{1}_{Z'} - \mathbf{1}_Z)$ is in $\mathcal{K}(\Omega\times\Omega)$.

Similarly, if we seek $\xi^- \in \mathcal{H}'_{\mathfrak{r}}$ satisfying $\pi(g_0)\xi^- = -\xi^-$, we are led to the conditions

$$y_k = -\frac{s_1s_2}{q}x_{k+1} \quad \text{for } k \geq 0, \quad \text{and} \quad z' = -\frac{s_1}{s_2}z.$$

Again, the second of these conditions implies the first. So if we set $z = s_2$, $z' = -s_1$, and substitute these values of z, z' into the formulas (5.10) for the x_k's and y_k's, we get an element $\xi^- \in \mathcal{H}'_{\mathfrak{r}}$ such that $\pi(g_0)\xi^- = -\xi^-$. Notice that

$$y_n = \frac{(s_1-s_2)}{(q-1)(s_1s_2)^n} \quad \text{for } n \geq 0,$$

and so in the case $s_1 = s_2$, $\xi^- = s_1(\mathbf{1}_Z - \mathbf{1}_{Z'})$ is in $\mathcal{K}(\Omega\times\Omega)$.

If we let $u(g) = \langle \pi(g)\xi^+, \xi^-\rangle_{s_1,s_2}$, then u is bi-$K(\mathfrak{r})$-invariant and satisfies (2.14), and so determined by $u(1)$ and $u(g_0)$ (see the proof of Lemma 2.2). Now

$$u(g_0) = \langle \pi(g_0)\xi^+, \xi^-\rangle_{s_1,s_2} = \langle \xi^+, \xi^-\rangle_{s_1,s_2} = u(1),$$

and also

$$u(g_0) = \langle \pi(g_0)\xi^+, \xi^-\rangle_{s_1,s_2} = \langle \xi^+, \pi(g_0^{-1})\xi^-\rangle_{s_1,s_2} = -\langle \xi^+, \xi^-\rangle_{s_1,s_2} = -u(1).$$

It follows that $u = 0$. Hence the linear span $\mathcal{H}_+$ of $\{\pi(g)\xi^+ : g \in G\}$ is orthogonal to the linear span $\mathcal{H}_-$ of $\{\pi(g)\xi^- : g \in G\}$. Clearly $\mathcal{H}_{\text{sp}} = \mathcal{H}_+ \oplus \mathcal{H}_-$. We also know that $\mathcal{H}_{\text{sp}} \subset H_2$, by Lemma 2.2. Moreover, $\mathcal{H}_{\text{sp}}$ can contain no nonzero $\xi \in \mathcal{H}'_{\mathfrak{r}'}$ for any $\mathfrak{r}'$ as in Figure 1(b) or Figure 1(c_r). For if $\eta \in \mathcal{H}_{\text{sp}}$, then $u(g) = \langle \pi(g)\xi, \eta\rangle_{s_1,s_2}$ would be right $K(\mathfrak{r}')$-invariant, left $K(\mathfrak{r})$-invariant, and satisfy (2.14) (with $\mathfrak{r}$ there replaced by $\mathfrak{r}'$), and by [2, Proposition III.3.2] would therefore be supported in the empty set $\{g \in G : g\mathfrak{r}' \subset \mathfrak{r}\}$.

Hence the restrictions of π to $\mathcal{H}_+$ and $\mathcal{H}_-$ are equivalent to special representations of G. They are inequivalent, for if $T : \mathcal{H}_+ \to \mathcal{H}_-$ intertwines these two restrictions, then $\xi = T\xi^+$ satisfies $\pi(g_0)\xi = \xi$, and so ξ is a multiple of ξ^+. But $\mathcal{H}_+$ and $\mathcal{H}_-$ are orthogonal, and so $\xi = 0$, and therefore $T = 0$. □

References

1. D.I. Cartwright, G. Kuhn and P.M. Soardi, *A product formula for spherical representations of a group of automorphisms of a homogeneous tree, I*, Trans. Amer. Math. Soc. **353**, 2000, 349–364. CMP 99:17
2. A. Figà-Talamanca and C. Nebbia, *Harmonic analysis and representation theory for groups acting on homogeneous trees*, London Mathematical Society Lecture Note Series **162**, Cambridge University Press, Cambridge, 1991. MR **93f:**22004
3. G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of mathematics and its applications, Volume 16, Addison-Wesley Publishing Company, 1981. MR **83k:**20003
4. G.I. Olshanskii, *Classification of irreducible representations of groups of automorphisms of Bruhat-Tits trees*, Functional Anal. Appl., **11**, 1977, 26–34.
5. J. Repka, *Tensor products of unitary representations of $SL_2(\mathbb{R})$*, Amer. J. Math., **100**, 1978, 747–774. MR **80g:**22014

School of Mathematics and Statistics, University of Sydney, N.S.W. 2006, Australia
E-mail address: donaldc@maths.usyd.edu.au

Dipartimento di matematica e applicazioni, Università di Milano-Bicocca, Viale Sarca 202, Edificio U7, 20126 Milano, Italy
E-mail address: kuhn@vmimat.mat.unimi.it